1. (a)

$$[B|\vec{y}] = A \xrightarrow{RREF} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 3 & 7/3 \\ 0 & 1 & 0 & 7 & 1 \\ 0 & 0 & 1 & 1 & -1/2 \end{array}\right]$$

So
$$x_1 = \frac{7}{3} - 3x_4$$
$$x_2 = 1 - 7x_4$$
$$x_3 = -\frac{1}{2} - x_4$$
x_4 free.

In parametric vector form, we have

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 7/3 \\ 1 \\ -1/2 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -3 \\ -7 \\ -1 \\ 1 \end{pmatrix}.$$

Because there is a free variable, $\mathcal{L}$ does not have a unique solution.

(b)

$$C \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The equation $C\vec{x} = \vec{0}$ does not have a nontrivial solution as each column of C contains a pivot. In particular, the columns of C are linearly independent.

(c) (i) False.

(If $\vec{b} \neq \vec{0}$, the system can be inconsistent.)

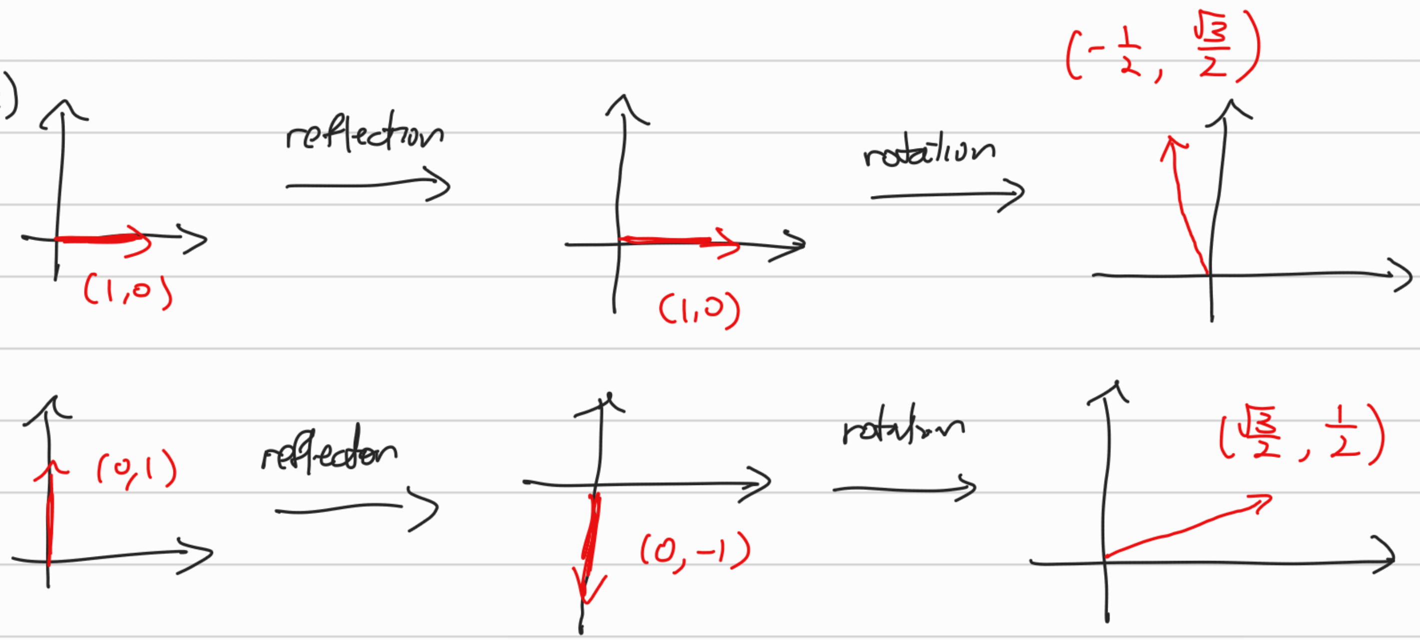
(ii) False.

(Four lin. indep. columns in $\mathbb{R}^4$ will automatically span $\mathbb{R}^4$.)

(iii) True.

(U is row equivalent to the identity.)

2. (a)



So $T(1,0) = (-\frac{1}{2}, \frac{\sqrt{3}}{2})$

$T(0,1) = (\frac{\sqrt{3}}{2}, \frac{1}{2})$.

So
$$A = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

Then $\det(A) = -\frac{1}{4} - \frac{3}{4} = -1$, and so

$$\det(A^2) = (\det(A))^2 = 1.$$

(b) As A is a square matrix with nonzero determinant, it must have pivots in each column and row. Hence, T is both one-to-one and onto.

(c) $U: \mathbb{R}^4 \to \mathbb{R}^5$ where $U(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, x_4, 0)$.

(d) (i) False.

(e.g. $T: \mathbb{R}^2 \to \mathbb{R}$ where $T(x,y) = x$ is onto but not 1-1.)

(ii) False.

(iii) True.

3. (a) Area of $S = \left| \det \begin{pmatrix} -1 & 5 \\ 3 & -5 \end{pmatrix} \right|$

$= |5 - 15|$

$= 10.$

Area of $T(S) = |\det(T)| \cdot (\text{area of } S)$

$= 10 \cdot \left| \det \begin{pmatrix} 2 & -2 \\ 7 & -3 \end{pmatrix} \right|$

$= 10 \cdot |-6 + 14|$

$= \underline{80}.$

(b) $|B| = \begin{vmatrix} 8 & 3 \\ 0 & 6 \end{vmatrix} - 5 \begin{vmatrix} 3 & 3 \\ -1 & 6 \end{vmatrix}$

$= 48 - 15 \begin{vmatrix} 1 & 1 \\ -1 & 6 \end{vmatrix}$

$= 48 - 15 \times 7$

$= -57$

So $\det(B^{-1}) = -\frac{1}{57}$.

(c) (i) True

(If $A = P^{-1}BP$, then $|A| = |P^{-1}||B||P| = |B|$.)

(ii) False.

$(|7C| = 7^5|C|.)$

4. (a) (i) Basis of Row A is

$\{(1,0,0,3,7)^T, (0,1,0,-6,-4)^T, (0,0,1,0,2)^T\}$.

Dim. of Row A is 3.

(ii) Basis of Col A is

$\{(0,-1,-1,-1,3), (0,0,0,1,5), (1,-5,-4,-1,3)\}$.

Dim. of Col(A) is 3.

(iii) Basis of Nul A is

$\{(-3,6,0,1,0), (-7,4,-2,0,1)\}$

Dim of Nul A is 2.

(iv) $\text{rank}(A) = \dim(\text{Row } A) = 3$

$\text{nullity}(A) = \dim(\text{Nul } A) = 2$

# of columns of $A = 5$.

Rank-nullity: $\text{rank}(A) + \text{nullity}(A) = \#$ of columns of A

$\rightsquigarrow 3 + 2 = 5$

(b) $3\vec{a}_1 - 6\vec{a}_2 - \vec{a}_4 = 0.$

5. (a) (i) $[3t^2 - 4t + 9] = (9, -4, 3)$

(ii) Note that

$$\underset{S \leftarrow B}{P} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$$

Then $\underset{B \leftarrow S}{P} = \left(\underset{S \leftarrow B}{P}\right)^{-1}.$

Note that

$$\left[\underset{S \leftarrow B}{P} \;\middle|\; I\right] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{RREF} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

So $$\underset{B \leftarrow S}{P} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}.$$

(iii) $$[3 - 5t]_B = \underset{B \leftarrow S}{P}\, [3 - 5t]_S$$

$$= \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ -5 \\ 0 \end{bmatrix}$$

$$= \begin{pmatrix} -200 \\ 64 \\ 25 \end{pmatrix}$$

(b) (i) False.

(ii) True.

($\text{rank}(A) = \dim(\text{Row} A) \leq 5$)

(iii) False.

6. (a) (i) $p_A(\lambda) = \det(A - \lambda I)$

$$= \det \begin{bmatrix} 2-\lambda & -3 & -4 \\ 0 & -\lambda & -2 \\ 0 & 1 & 3-\lambda \end{bmatrix}$$

$$= (2-\lambda) \begin{vmatrix} -\lambda & -2 \\ 1 & 3-\lambda \end{vmatrix}$$

$$= (2-\lambda)(\lambda^2 - 3\lambda + 2)$$

$$= (2-\lambda)(\lambda-2)(\lambda-1).$$

So the eigenvalues are 1, 2.

(ii) Note that $\lambda = 1$ has multiplicity 1 and $\lambda=2$ has multiplicity 2.

Now

$$A - 2I = \begin{pmatrix} 0 & -3 & -4 \\ 0 & -2 & -2 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 1 \end{pmatrix}$$

$$\to \begin{pmatrix} 0 & 1 & 1 \\ 0 & 3 & 4 \\ 0 & 0 & 0 \end{pmatrix}$$

has rank 2, so it has nullity 1. So the eigenspace for $\lambda = 2$ is 1-dimensional, so A is not diagonalizable.

(iii) No.

(b) (i) False.

(ii) False

7. (a) (i) Let $\vec{w}_1 = (2,2,-1,3)$, $\vec{w}_2 = (1,9,2,6)$.

Then

$$\text{proj}_{\vec{w}_1}(\vec{w}_2) = \frac{\vec{w}_1 \cdot \vec{w}_2}{\vec{w}_1 \cdot \vec{w}_1} \vec{w}_1$$

$$= \frac{36}{18} \vec{w}_1$$

$$= (4,4,-2,6).$$

Then

$$\vec{w}_2 - \text{proj}_{\vec{w}_1}(\vec{w}_2) = (-3,5,4,0)$$

Let $\vec{v}_1 = \vec{w}_1$, $\vec{v}_2 = (-3,5,4,0)$.

Then $\frac{\vec{v}_1}{\|\vec{v}_1\|} = \frac{1}{\sqrt{18}}(2,2,-1,3) = \frac{1}{3\sqrt{2}}(2,2,-1,3)$

and $\frac{\vec{v}_2}{\|\vec{v}_2\|} = \frac{1}{\sqrt{50}}(-3,5,4,0) = \frac{1}{5\sqrt{2}}(-3,5,4,0)$

form an orthonormal basis for W

(ii) Note that

$$W^\perp = (\mathrm{Col}\,A)^\perp$$
$$= \mathrm{Nul}\,A^T$$
$$= \mathrm{Nul}\begin{pmatrix} 2 & 2 & -1 & 3 \\ 1 & 9 & 2 & 6 \end{pmatrix}$$
$$\overset{\text{RREF}}{=} \mathrm{Nul}\begin{pmatrix} 1 & 0 & -\frac{13}{16} & \frac{15}{16} \\ 0 & 1 & \frac{5}{16} & \frac{9}{16} \end{pmatrix}$$

$$= \mathrm{Span}\{(\tfrac{13}{16}, -\tfrac{5}{16}, 1, 0), (-\tfrac{15}{16}, -\tfrac{9}{16}, 0, 1)\}$$

So $\{(13,-5,16,0), (-15,-9,0,16)\}$ is a basis for $W^\perp$.

(b) Plugging in the four points gives

$$a+b+c = 2$$
$$a-b+c = 3$$
$$16a+4b+c = 1$$
$$25a+5b+c = 2$$
$$49a+7b+c = -3.$$

So the matrix equation is

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 16 & 4 & 1 \\ 25 & 5 & 1 \\ 49 & 7 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 2 \\ -3 \end{bmatrix}.$$